A Diophantine equation
(corrected slides)

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The equation.

In Vassil Dimitrov's talk this afternoon, the following statement was presented as a conjecture.

Theorem

For all non-negative integers a, b, c, d, e, f we have

$$\pm 2^a 3^b \pm 2^c 3^d \pm 2^e 3^f \neq 4985.$$

The equation.

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Theorem

For all non-negative integers a, b, c, d, e, f *we have*

$$\pm 2^a 3^b \pm 2^c 3^d \pm 2^e 3^f \neq 4985.$$

The theorem is equivalent to:

Theorem

For all non-negative integers a, b, c, d, e, f *we have*

$$\pm 1 \pm 2^c 3^d \pm 2^e 3^f \neq 4985$$

$$\pm 2^a \pm 3^d \pm 2^e 3^f \neq 4985$$

Proof.

$$\begin{aligned} \text{Let } n &= \gcd(2^{180}-1, 3^{180}-1) \\ &= 439564261361225 \\ &= 5^2 \cdot 7 \cdot 11 \cdot 13 \cdot 19 \cdot 31 \cdot 37 \cdot 61 \cdot 73 \cdot 181 \end{aligned}$$

Claim: For all integers a, b, c, d, e, f we have

$$\pm 1 \pm 2^c 3^d \pm 2^e 3^f \not\equiv 4985 \bmod n$$
$$\pm 2^a \pm 3^d \pm 2^e 3^f \not\equiv 4985 \bmod n$$

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This is a finite computation, and so is trivial.

No, really! It's not hard.

Let

$$S = \{\pm 2^a \cdot 3^b : a, b \in \mathbb{Z}\} \subset (\mathbb{Z}/n\mathbb{Z})$$
$$T = \{\pm 2^a \pm 3^b : a, b \in \mathbb{Z}\} \subset (\mathbb{Z}/n\mathbb{Z})$$

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Then $\#S = 64800$ and $\#T = 129543$.

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Then $\#S = 64800$ and $\#T = 129543$.

For t in $\{1, -1\}$ compute the intersection

$$S \cap \{s + t + 4985 : s \in S\},$$

and also compute the intersection

$$S \cap \{t + 4985 : t \in T\}.$$

All three are empty.

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This proves the claim.