Fallibility and other real-life problems

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Conference on Open Questions in Cryptography and Number Theory
UC Irvine, 17–21 September 2018

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Lots of people turn 60, but Alice has done more than that. . .

[Explanatory note: Here is where I mentioned Alice's contributions to mathematics and her support of more junior people — including me, over the years — and wished her a happy birthday.]

Part 1: Euler

An unexpected quotation

Ralph Waldo Emerson, in *Nature* (1836)

The astronomer, the geometer, rely on their irrefragable analysis, and disdain the results of observation. The sublime remark of Euler on his law of arches, “This will be found contrary to all experience, yet is true;” had already transferred nature into the mind, and left matter like an outcast corpse.

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- If so, why?

Almost certainly, Emerson was not quoting Euler directly.

Samuel Taylor Coleridge, in *Aids to Reflection* (1825)

The celebrated Euler having demonstrated certain properties of Arches, adds: "All experience is in contradiction to this; but this is no reason for doubting its truth." The words sound paradoxical; but mean no more than this [. . .]

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Enlightenment!

I doubt Coleridge read Euler. But I bet he read Voltaire.

Voltaire, in *Diatribe du docteur Akakia* (1752)

He [Euler] *asks forgiveness, on his knees, from all logicians for having written, on the occasion of a result contradicting his calculations: "Indeed, this [calculation] appears to be less in agreement with the truth; [. . .] In any case, the calculation rather than our judgement is to be trusted."*

Voltaire and Euler

Voltaire wrote the satirical *Diatribe du docteur Akakia* because he and Euler were on opposite sides of a contretemps in the court of Frederick the Great.

See: Paul Nahin, *Dr. Euler's fabulous formula* (Princeton University Press, 2006).

Euler: *Mechanica* (1736), Chapter 3, paragraph 272

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What is the context?

- Studying linear motion modeled by $\frac{d^2x}{dt^2} = kx^n$.
- What happens when $x = 0$? (n may be negative.)
- What happens when $x < 0$? (n may not be an integer.)
- For $n \leq -1$, Euler says the object will attain infinite speed as $x \to 0^+$...
- ...and then immediately reverse course once $x = 0$.

See: Eberhard Knobloch, "Euler and infinite speed", *Doc. Math.* 2012 (extra vol.)

Mathematical morals

What to take away from this story?

- In this case, Euler's model was not great.
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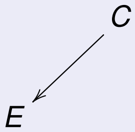
[This point of view is certainly not new, or original to me. For an application to current events, see Cathy O'Neil's comments on the ABC conjecture.]

Part 2: Sleight of hand

Genus-2 double covers of elliptic curves

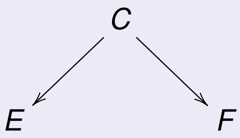
Given a genus-2 double cover of an elliptic curve $C \to E \ldots$



Genus-2 double covers of elliptic curves

Given a genus-2 double cover of an elliptic curve $C \to E$...

... there is a second double cover of an elliptic curve $C \to F$...



Background

Genus-2 double covers of elliptic curves

Given a genus-2 double cover of an elliptic curve $C \to E \ldots$
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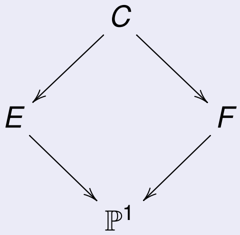
$\ldots$ so that both fit into a V_4 Galois extension of $\mathbb{P}^1$.



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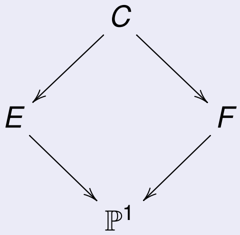

Old result: versions going back to Königsberger (J. Reine Angew. Math., 1867)

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A colleague at a university showed me a proof of something they were working on.

Something seemed “off” to me. No mistake jumped out, but. . .

I found a counterexample to a lemma.

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Them: But I have a proof.

Me: $\langle \ldots \rangle$

A beautiful error

A construction

Them: I have an elliptic curve E...

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Them: I have an elliptic curve E. . .

. . . and I have constructed a genus-2 double cover $C \to E$.

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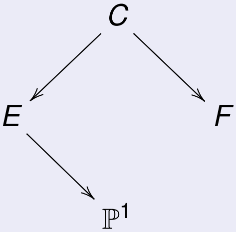

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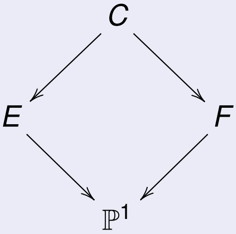

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My takeaway

- Errors can be subtle!
- It can be very hard to spot our own errors.
- We *need other people* to look at our proofs.

Part 3: An error in the wild

Background

Jeff Achter: What are some properties of principally-polarized abelian surfaces over finite fields that are rare, but not too rare?

If C is a genus-2 curve over $\mathbb{Q}$ we can study

$$\pi_T(C,x) = \#\left\{ p \le x : \begin{array}{l} \text{Jac } C \text{ has good reduction mod } p \\ \text{and the reduction has property } T \end{array} \right\}.$$

Let $P_T(q)$ be the probability that a randomly-chosen principally-polarized abelian surface over $\mathbb{F}_q$ has property T.

Naïvely expect $\pi_T(C,x) \sim \sum_{p<x} P_T(p)$.

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If $P_T(q) > c/q$, we expect $\pi_T(C,x)$ to grow visibly over ranges of x where we can actually compute it.

Instructive non-example

Consider the property T of *not being a Jacobian*.

Can show that $P_{\mathrm{nonJac}}(q) \approx 1/q$.

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This example demonstrates:

- Why we would like properties T with $P_T(q) > c/q$, and
- That the naïve view is naïve.

Our choice

We looked at the property T of being *split over* $\mathbb{F}_q$.

Theorem (Achter-H. 2017)

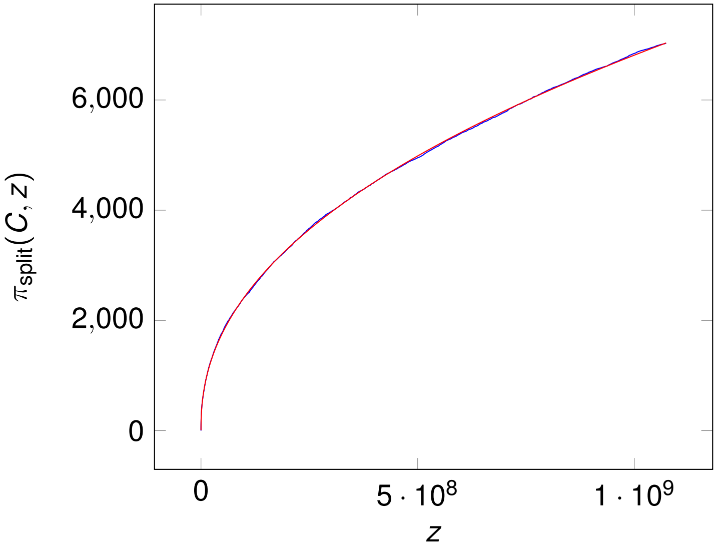
For all q we have

$$\frac{1}{(\log q)^3(\log\log q)^6} \ll P_{\text{split}}(q)\sqrt{q} \ll (\log q)^4(\log\log q)^2.$$

Conjecture (Achter-H. 2017)

Let J be the Jacobian of a genus-2 curve over $\mathbb{Q}$ with $\operatorname{End} J = \mathbb{Z}$. Then there is a constant $c_J > 0$ such that

$$\pi_{\text{split}}(J, x) \sim c_J \frac{\sqrt{x}}{\log x} \quad \textit{as } x \to \infty.$$



The blue curve is $\pi_{\text{split}}(C, z)$ for $C\colon\ y^2 = x^5 + x + 6$.

The red curve is $c\sqrt{z}/\log z$ with $c \approx 4.4651$.

[Here I thanked Drew Sutherland verbally — he provided programs and computer time to help produce this data.]

Open questions

1. What are interesting properties T of principally-polarized abelian surfaces with $P_T(q) > c/q$?
2. What are some approaches to proving the conjecture?
3. We look at surfaces that are split over $\mathbb{F}_q$. What about surfaces that are geometrically split?

Moduli spaces of supersingular genus-2 curves

The theorem above gives bounds for the number of non-simple principally-polarized abelian surfaces over $\mathbb{F}_q$.

A sub-problem

- Understand the moduli space of supersingular genus-2 curves.
- Moduli space of genus-2 curves: Subvariety of weighted projective space $[J_2 : J_4 : J_6 : J_8 : J_{10}]$.
- $J_{10} \neq 0$, plus single relation: $J_2 J_6 = J_4^2 + 4J_8$.

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I wanted to look at case $q = 5$.

Computing equations for supersingular locus

Quick and dirty method

- Compute many supersingular genus-2 curves over $\mathbb{F}_{5^n}$.
- Compute their Igusa invariants $[J_2 : J_4 : J_6 : J_8 : J_{10}]$.
- Find low-degree polynomials vanishing at these points.
- Can prove guesses later.

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Manin and Yui

- Curve $C\colon y^2 = f = a_6x^6 + \cdots + a_0$ over $\mathbb{F}_{5^n}$.
- Let $g = f^2 = b_{12}x^{12} + \ldots + b_0$.
- Set $M = \begin{bmatrix} b_4 & b_3 \\ b_9 & b_8 \end{bmatrix}$ and $M^{(5)} = \begin{bmatrix} b_4^5 & b_3^5 \\ b_9^5 & b_8^5 \end{bmatrix}$.
- Yui, Manin: C is supersingular if and only if $MM^{(5)} = 0$.

Equations for the supersingular locus?

- Fit polynomials to many supersingular Igusa invariants in characteristic 5.
- In addition to $J_2 J_6 = J_4^2 + 4J_8$, found:
  - $J_4 = -J_2^2$.
  - Polynomial of weighted degree 1050 in J_2, J_6, and J_{10}.
- Highly singular subvariety of $\mathcal{M}_2$.
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- Had we misunderstood the result of Koblitz?
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- We computed the zeta function of one of the curves that the Yui/Manin condition said was supersingular.
- It wasn't supersingular.

“Sign errors”

Manin (1961) and Yui (1978)

- Given a genus-g hyperelliptic curve $y^2 = f$ over $\mathbb{F}_{p^e}$, with Jacobian J.
- Let σ be the p-power Frobenius. There is a matrix M such that
  - the p-rank of J is the p-rank of $MM^{(\sigma)}M^{(\sigma^2)}\cdots M^{(\sigma^{g-1})}$.
  - the characteristic polynomial of p^e-Frobenius on J is related to the characteristic polynomial of $MM^{(\sigma)}M^{(\sigma^2)}\cdots M^{(\sigma^{e-1})}$.
- Yui gives a formula for M in terms of coefficients of $f^{(p-1)/2}$.

Yui’s formula for *M* does not work with these results. Need to:

- transpose M, or
- multiply the matrices in the opposite order, or
- replace σ with p-th root automorphism.

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- Yui gives a formula for M in terms of coefficients of $f^{(p-1)/2}$.

Yui’s formula for *M* does not work with these results. Need to:

- transpose M, or
- multiply the matrices in the opposite order, or
- replace σ with p-th root automorphism.

Note: For examples in Yui (1978), all M are *diagonal*, so mistake is not apparent.

Consequences

Equations for supersingular locus in $\mathcal{M}_2$ over $\mathbb{F}_{5^n}$

- Look at curves with $M^{(5)}M = 0$.
- In addition to $J_2J_6 = J_4^2 + 4J_8$, find:
  - $J_4 = -J_2^2$
  - $J_{10}J_2 = 3J_6^2 + 2J_6J_2^3 + 2J_2^6$
- Subvariety is image of $[2t^2 : t^4 : 3t + 4t^6 : 4t^3 + 3t^8 : 1]$.

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Less pleasant consequences

- I *used* the incorrect criterion in an earlier paper!
- Panic.
- With the correct criterion, my proof still went through. Whew!

Ramifications

The results from *my* earlier paper still stood.

What about *other* papers that used the criterion?

Jeff and I decided to track down papers that cited Yui (1978) or Manin (1961).

Surely that wouldn't be *too* many papers. . .

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A total of 91 papers

Many papers were still OK

- Cited Manin and Yui as general references, no results quoted.
- Or, results *were* quoted, but:
  - The quoted results were not applied.
  - The quoted results did not contain errors.
  - Errors were silently corrected.
- Or, incorrect results *were* applied, but the formulas worked anyway:
  - The matrix M was diagonal.
  - M had entries in $\mathbb{F}_p$ or $\mathbb{F}_{p^2}$, so $M^\sigma = M^{\sigma^{-1}}$.
  - ...

But eight papers required a closer look.

Results

We contacted Manin, Yui, and the authors of the eight papers.

They were generous, polite, and glad to learn of the problem.

See the paper that Jeff and I wrote about this for more details.

My takeaway

- Mistakes can sit unnoticed for a long time.
- Mistakes can spread.

Part 4: Alice in the real world

Lessons from mistakes

- Everyone makes mistakes.
- They can be hard to see ourselves.
- They can remain there for years.
- Their effects can be widespread.
- To find them and fix them, we need to
    - listen to criticism;
    - think differently than we may be used to;
    - not get defensive.

Alice's adventures in numberland

For about a year now, Alice has been blogging about stories from her life.

To me, some of these stories seem to be about mistakes:
Mistakes by mathematicians, but not about mathematics.

Among the many experiences she writes of, some tell of mistakes — *injustices* is a better word — about how women are treated in our profession.

A gift

For some of us, Alice's blog is the gift of a different perspective.

In my life, I've seen some of these injustices firsthand, from the low-level everyday ones to the rare and extraordinary ones. [Some stories were mentioned here.]

But I haven't seen them all.

- Sometimes, I am not in the room to see.
- Sometimes, I am present, but do not notice.
- Sometimes, I make mistakes myself, despite my best intentions.

Alice's blog helps me be aware of problems I might otherwise miss —
and there are still *more* voices to listen to, because sexism is not the only problem.

A referee report

As mathematicians, we value:

- the colleagues who read and comment on our preprints;
- the referees who check our reasoning and catch mistakes;
- the editors who improve our writing.

We should also value the people who tell stories of how our community fails to live up to our highest values.

Open question, closing hope

An open question:

How do we want the community of mathematicians to be experienced — by all of its members?

A closing hope

May we hear and appreciate criticisms of our community...
May we acknowledge the injustices that occur in our profession...
And may we *actively do something to remedy them*.